Analytic solutions of vortex lattice on hyperbolic plane

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1. Introduction

2. vortex on compact Riemann surface

3. ”vortex lattice” on hyperbolic plane

4. summary
1. Introduction
1.1 Bogomolny equations

Vortex: a soliton solution of the Abelian-Higgs model

Energy function (at critical coupling) is saturated by Bogomolny bound.

\[ E = \frac{1}{2} \int \left( \frac{B^2}{2\Omega} + |D_i \phi|^2 + \frac{\Omega}{2}(1 - |\phi|^2)^2 \right) dx dy \geq \pi |N|. \] integer N: vortex number

\[ B = f_{12} = \partial_1 a_2 - \partial_2 a_1, \quad D_i = \partial_i - i a_i \]

2D Constant curvature surfaces: \( ds^2 = \Omega dz d\bar{z} \)

\[ S^2 \text{ (constant positive)}, \quad R^2 \text{ (curvature 0)}, \quad H^2 \text{ (constant negative)} \]

The minimum energy condition is given by the following equations:

Bogomolny equations \( \bar{D}_z \phi = 0, \quad B = \Omega \left( 1 - |\phi|^2 \right) \)

Vortex: the Higgs field takes a VEV at infinity \( |\phi| \to 1, \quad D_i \phi \to 0 \).
1.2 Vortex equations on hyperbolic plane

**hyperbolic plane**

**Gauss curvature** \( K = -\frac{1}{2\Omega} \nabla^2 \log(\Omega) = -1 \)

**Poincaré disk**

\[ ds^2 = \frac{4}{(1 - |z|^2)^2} dz d\bar{z} \]

**Upper half plane**

\[ ds^2 = \frac{d\zeta d\bar{\zeta}}{(\text{Im } \zeta)^2} \]

On the Poincaré disk, we eliminate the gauge field, and the Bogomolny equation becomes,

**Taubes equation**

\[ 4 \partial_z \partial_{\bar{z}} \log |\phi|^2 + 2\Omega(1 - |\phi|^2) = 0 \]
### Solutions to Taubes equation

\[ 4 \partial_z \partial_{\bar{z}} \log |\phi|^2 + 2\Omega(1 - |\phi|^2) = 0 \]

**using the solution to Liouville equation**

\[
\phi(z, \bar{z}) = \frac{1 - |z|^2}{1 - |f(z)|^2} \frac{df}{dz}
\]

It is given by arbitrary holomorphic function \( f(z) \).

\[
|\phi(z, \bar{z})|^2 = \frac{\tilde{\Omega}(f(z))}{\Omega(z)} \left| \frac{df}{dz} \right|^2
\]


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**The case of finite number of vortex**:

Blaschke product

\[
f(z) = \prod_{j=1}^{N+1} \frac{z - a_j}{1 - \bar{a}_j z}, \quad |a_j| < 1
\]

\[ N = 1, \quad a_1 = 0, \quad a_2 = 0 \quad f(z) = z^2 \quad \phi(z, \bar{z}) = \frac{2z}{1 + |z|^2}
\]

- \( |z| \to 0 \quad |\phi| = 0 \)
  - the center of the vortex
- \( |z| \to 1 \quad |\phi| = 1 \)
  - a VEV at the circumference

1-vortex at the origin

**VEV**
They succeeded in constructing the exact vortex solution for the Taubes equation on the compact Riemann surface.

the compact Riemann surfaces with two or more Genus can be constructed from hyperbolic plane with appropriate identification:

- Bolza surface
- \( g=2 \) compact Riemann surface

- \{8,8\} tessellation of Poincaré disk.
2.1 Hyperbolic plane and compact Riemann surface

\{p,q\} tessellation of Poincare disk

At each vertex, the fundamental polygon, \("p\)-gon\", meets other \(q\) identical polygons.

Ex) \{8,8\}

- congruent regular octagons.
- eight octagons are meeting at each vertex.

\[ \frac{2p}{p - 2} < q \]

\{8,5\}

\{5,10\}

\{12,7\}
They succeeded in constructing the exact vortex solution for the Taubes equation on the compact Riemann surface.

The compact Riemann surfaces with two or more Genus can be constructed from hyperbolic plane with appropriate identification.

- \{8,8\} tessellation of Poincaré disk.
- Identifying the opposite sides of octagons.
2.2 1-vortex on compact Riemann surface

\( g = 2 \) compact Riemann surface can be regarded as double cover of Riemann spheres with six branch points.

Put the vortex at branch point \( \rightarrow \) excess angle \( 2\pi \)

\[ w = f(z) \]
2.3 Regular function which give 1-vortex on compact Riemann surface

Riemann mapping: $\exists$ a map which maps the entire upper half plane into an arbitrary polygon

Schwarz triangle map: mapping from the upper half plane to triangle on the Poincaré disk

$w = f(z) = s_2(s_1^{-1}(z))$
2.4 Schwarz triangle map

Schwarz triangle map $s$ : given by a ratio of solutions to Hypergeometric differential eq.

Hypergeometric differential equation:
\[
\frac{d^2 y}{d\zeta^2} + \left( \frac{c}{\zeta} + \frac{d}{\zeta - 1} \right) \frac{dy}{d\zeta} + \frac{ab}{\zeta(\zeta - 1)} y = 0
\]

Solutions:
\[
y(\zeta) = F(a, b, c; \zeta), \quad \tilde{y}(\zeta) = \zeta^{1-c} F(a', b', c'; \zeta)
\]

Parameters:
\[a, b, c, d\] are determined by the interior angle of the triangle.

\[
s(a, b, c; \zeta) = \mathcal{N} \frac{\tilde{y}(\zeta)}{y(\zeta)} = \sqrt{\frac{\sin(\pi a') \sin(\pi b') \Gamma(a')\Gamma(b')\Gamma(c')}{\sin(\pi a) \sin(\pi b) \Gamma(a)\Gamma(b)\Gamma(c)}} \frac{\zeta^{1-c} F(a', b', c'; \zeta)}{F(a, b, c; \zeta)}
\]

three definite singularities of the hypergeometric function correspond to vertices of the triangle.
\[A \leftrightarrow 0, \quad B \leftrightarrow 1, \quad C \leftrightarrow i\infty\]
We show that the Higgs field values at the vertices of the triangle can be determined.

Near $\zeta = 0$

$$s(\zeta) = \sqrt{\frac{\sin(\pi a') \sin(\pi b')}{\sin(\pi a) \sin(\pi b)} \frac{\Gamma(a') \Gamma(b') \Gamma(c')}{\Gamma(a) \Gamma(b) \Gamma(c)}} \zeta^a + \cdots$$

$$z = s_1(\zeta) = \mathcal{N}_1 \zeta^{\frac{1}{8}} + \cdots$$

$$s_2 = (\zeta) = \mathcal{N}_2 \zeta^{\frac{1}{4}} + \cdots$$

$$\phi = \frac{\mathcal{N}_2}{\mathcal{N}_1^2} z + \cdots$$

Near $\zeta = 1$

$$s(1 + \zeta) = \sqrt{\frac{\sin(\pi a) \sin(\pi b)}{\sin(\pi a') \sin(\pi b')} \left(1 - \zeta^\beta \frac{\sin(\alpha \pi)}{\pi \beta \Gamma^2(\beta)} \Gamma(1 - a) \Gamma(1 - b) \Gamma(1 - a') \Gamma(1 - b') + \cdots\right)}$$

$$s_1(1 + \zeta) = \mathcal{A}_1 + \mathcal{B}_1 \zeta^{\frac{1}{2}} + \cdots$$

$$s_2(1 + \zeta) = \mathcal{A}_2 + \mathcal{B}_2 \zeta^{\frac{1}{2}} + \cdots$$

$$\phi = \frac{1 - \mathcal{A}_1^2 \mathcal{B}_2}{1 - \mathcal{A}_2^2 \mathcal{B}_1}$$

$$|\phi|_{\text{origin}} = \left[(4\pi)^{-\frac{3}{2}} \sin\left(\frac{\pi}{8}\right) \Gamma^2\left(\frac{1}{8}\right) \Gamma\left(\frac{1}{4}\right)\right] + O(|z|^3) \approx 1.768|z| + O(|z|^3)$$

$$|\phi|_{\text{mid edge}} = \frac{\sqrt{2} \Gamma\left(\frac{1}{8}\right) \Gamma^2\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{8}\right)}{\Gamma\left(\frac{1}{16}\right) \Gamma\left(\frac{3}{16}\right) \Gamma\left(\frac{5}{16}\right) \Gamma\left(\frac{7}{16}\right)} \approx 0.752$$

$$|\phi|_{\text{vertex}} = 2^{-\frac{1}{4}} \approx 0.841$$

they could calculate the Higgs field at the vertices of the triangle.
Chapter 3: Vortex lattices on hyperbolic planes (not compact Riemann surfaces)

3.1 Other tessellations.

3.1.1 changing polygon type.

3.1.2 changing meeting number.

3.2 Other excess angles.
3.1.1 changing the type of polygon (Hexagon → triangle)

\[ w = f(z) = s_2(s_1^{-1}(z)) \]

\[
|\phi|_{\text{origin}} = \frac{\sqrt{\frac{1}{2} + \frac{1}{\sqrt{2}} \Gamma \left(\frac{1}{6}\right) \Gamma \left(\frac{7}{6}\right) \Gamma \left(\frac{5}{24}\right) \Gamma \left(\frac{11}{24}\right)}}{4\pi^2} |z| + O(|z|^3) \approx 1.221|z| + O(|z|^3)
\]

\[
|\phi|_{\text{mid edge}} \approx 0.462
\]

\[
|\phi|_{\text{vertex}} \approx 0.577
\]
3.1.2 the case of different meeting number at each vertex

\[ w = f(z) = s_2(s_1^{-1}(z)) \]

\[ \psi \]

\[ \psi \]

The differential coefficients

|\(\Phi|\) Value of Higgs field

vertex

Approach 1.

Mid edge
3.2 Other excess angles

$w = f(z) = s_2(s_1^{-1}(z))$

- **$s_1(\zeta)$**
- **$s_2(\zeta)$**

Value of Higgs field

- ▲ $4\pi$
- ● $6\pi$

| $\Phi$ | Value of Higgs field $\blacktriangle 4\pi$, $\bullet 6\pi$

- vertex
- Approach 1.

- Mid edge
Summary

• vortex on the compact Riemann surface

we introduced that the Higgs field at the vertex of a triangle can be calculated.


• Vortex lattice on hyperbolic plane

In some examples, we could calculate the Higgs field at the vertex of the triangle.

Outlook

• “Ideal triangle” has an internal angle of 0.

• Elliptical modular function, Chazy equations…

“Ideal triangle”