Doubled Aspects of Vaisman Algebroid in Para-Hermitian Geometry

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Doubled Aspects of Vaisman Algebroid in Para-Hermitian Geometry

in collaboration with Haruka Mori and Shin Sasaki
based on arXiv:1901.04777

Workshop on Recent Developments in Mathematical Physics @ Osaka Pref. U.
to give a geometric realization of the Vaisman algebroid as a double of a pair of Lie algebroids
The outline of this talk

Introduction

Para-Hermitian Geometry

Doubled Aspects of Vaisman Algebroid in DFT

Discussion on DFT Gauge Symmetry

Summary
Introduction
Why Para-Hermitian Geometry?

— *Because, the geometry of Double Field Theory is described by a para-Hermitian geometry.*

[Vaisman '12]
Double Field Theory (DFT) [Hull-Zwiebach '09] = the T-duality covariantised an effective theory of strings

- a gravity theory defined on the doubled spacetime $\mathcal{M}^{2D}$
- $\mathcal{M}^{2D}$: locally given by $\mathcal{M}^{2D} \simeq M^D \times \tilde{M}^D$ ($M^D$: spacetime)
Double Field Theory (DFT) [Hull-Zwiebach '09] = the T-duality covariantised an effective theory of strings

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The Section Condition: needed to make DFT be a physical theory

$$\eta^{MN} \partial_M \ast \partial_N \ast = 0$$

- the physical spacetime: a $D$-dimensional space (not $2D$)
- S.C. is a constraint that reduces extra d.o.f.
- S.C. originally came from the LMC of closed strings
Double Field Theory (DFT) [Hull-Zwiebach '09]

= the T-duality covariantised an effective theory of strings

- a gravity theory defined on the doubled spacetime $\mathcal{M}^{2D}$
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The Section Condition: needed to make DFT be a physical theory

- S.C. is necessary due to the closedness of the DFT gauge algebra

$$[\hat{\mathcal{L}}_{\Xi_1}, \hat{\mathcal{L}}_{\Xi_2}] \approx \hat{\mathcal{L}}_{[\Xi_1, \Xi_2]} \mod \text{S.C.}$$
Double Field Theory (DFT) \cite{Hull-Zwiebach '09} = the T-duality covariantised an effective theory of strings

- a gravity theory defined on the doubled spacetime $\mathcal{M}^{2D}$
- $\mathcal{M}^{2D}$: locally given by $\mathcal{M}^{2D} \cong M^D \times \tilde{M}^D$ ($M^D$: spacetime)

- the metric and the $B$-field are on equal footing
- the generalised diffeomorphism in $\mathcal{M}^{2D}$: $\hat{\mathcal{L}}_\Xi$.
  - contains diffeo. in $M^D$ and $U(1)$ gauge transf. by 1-form
  - the difference between $\hat{\mathcal{L}}_\Xi$ and the ordinary Lie deriv. $\mathcal{L}_\Xi$

\[
\hat{\mathcal{L}}_\Xi \Phi^M = \mathcal{L}_\Xi \Phi^M + \Phi_K \partial^M \Xi^K
\]
Infinitesimal Gauge Symmetry in DFT

- Infinitesimal gen. gauge transf.: given by the gen. Lie deriv.
- Commutator of gen. Lie deriv.: governed by the C-bracket

\[ [\hat{L} \Xi_1, \hat{L} \Xi_2] \approx \hat{L}_{[\Xi_1, \Xi_2]} \]

- the C-bracket written by \( D \)-dim. quantities:

\[ [\Xi_1, \Xi_2]_C = [X_1, X_2]_L + \mathcal{L}_{X_1} \xi_2 - \mathcal{L}_{X_2} \xi_1 - \frac{1}{2} d(\iota_{X_1} \xi_2 - \iota_{X_2} \xi_1) \]

\[ + [\xi_1, \xi_2]_L + \tilde{\mathcal{L}}_{\xi_1} X_2 - \tilde{\mathcal{L}}_{\xi_2} X_1 + \frac{1}{2} \tilde{d}(\iota_{X_1} \xi_2 - \iota_{X_2} \xi_1) \]

where \( \Xi^M_i \partial_M = X^\mu_i \partial_\mu + \xi_{i, \mu} \tilde{\partial}^\mu \) is doubled vector field.
The mathematical structure corresponding to the C-bracket is the **Vaisman algebroid**. 

\[
[\Xi_1, \Xi_2]_C = [X_1, X_2]_L + \mathcal{L}_{X_1} \xi_2 - \mathcal{L}_{X_2} \xi_1 - \frac{1}{2} d(\iota_{X_1} \xi_2 - \iota_{X_2} \xi_1) \\
+ [\xi_1, \xi_2] \tilde{\mathcal{L}} + \tilde{\mathcal{L}}_{\xi_1} X_2 - \tilde{\mathcal{L}}_{\xi_2} X_1 + \frac{1}{2} \tilde{d}(\iota_{X_1} \xi_2 - \iota_{X_2} \xi_1)
\]
## Vaisman Algebroid

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| Courant algebroid is defined by $(\mathcal{C},[\cdot,\cdot],\rho_\mathcal{C},(\cdot,\cdot))$. | Vaisman algebroid is defined by $(\mathcal{V},[\cdot,\cdot]_\mathcal{V},\rho_\mathcal{V},(\cdot,\cdot))$.
| Courant bracket $[\cdot,\cdot]_\mathcal{C}$ satisfies the following axioms: | Vaisman bracket $[\cdot,\cdot]_\mathcal{V}$ satisfies the following axioms:
| C1. failure of Jacobi ident. | V1. Leibniz rule |
| C2. anchor: homomorphically maps | V2. compatibility b/w $\cdot,\cdot$ & $\rho_\mathcal{V}$ |
| C3. Leibniz rule | |
| C4. (section condition) | Comparing the two defs., only \textbf{C3} and \textbf{C5} are required for the Vaisman algebroid. |
| C5. compatibility b/w $(\cdot,\cdot)$ & $\rho_\mathcal{C}$ | |
**Vaisman Algebroid**

[Definition]

Courant algebroid is defined by 
\((C, [\cdot, \cdot], \rho_C, (\cdot, \cdot))\).

Courant bracket \([\cdot, \cdot]_C\) satisfies the following axioms:

**C1.** failure of Jacobi identity.

**C2.** anchor: homomorphic map

**C3.** Leibniz rule

**C4.** (section condition)

**C5.** compatibility b/w \((\cdot, \cdot)\) & \(\rho_C\)

[Definition]

Vaisman algebroid is defined by 
\((V, [\cdot, \cdot], \rho_V, (\cdot, \cdot))\).

Vaisman bracket \([\cdot, \cdot]_V\) satisfies the following axioms:

**V1.** Leibniz rule

**V2.** compatibility b/w \((\cdot, \cdot)\) & \(\rho_V\)

**C1, C2, and C4** need the **derivation condition**
A Courant algebroid can be constructed from a Lie bialgebroid.

[Liu-Weinstein-Xu '97]

The Lie bialgebroid \((L, L^*)\) requires a compatibility condition between a Lie algebroid \(L\) and a Lie coalgebroid \(L^*\).

The derivation condition of the Lie bialgebroid

= a compatibility cond. b/w a Lie algebroid and a Lie coalgebroid

\[ d_*[A, B]_S = [d_* A, B]_S + [A, d_* B]_S, \quad A, B \in \Gamma(L) \]
Vaisman Algebroid

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Given a Lie algebroid pair \((E, E^*)\), we consider \(\mathcal{V} = E \oplus E^*\).

- Non-degenerate bilinear form \((\langle \cdot, \cdot \rangle: \text{internal prod.}):\)
  \[
  (e_1, e_2)_\pm = \frac{1}{2} \left( \langle \xi_1, X_2 \rangle \pm \langle \xi_2, X_1 \rangle \right) \quad (X_i \in E, \xi_i \in E^*).
  \]

- Skew-symmetric bracket:
  \[
  [e_1, e_2]_\mathcal{V} = [X_1, X_2]_E + \mathcal{L}_{X_1} \xi_2 - \mathcal{L}_{X_2} \xi_1 + d(e_1, e_2)_- \\
  + [\xi_1, \xi_2]_{E^*} + \mathcal{L}_{\xi_1} X_2 - \mathcal{L}_{\xi_2} X_1 - d^*(e_1, e_2)_-.
  \]

- Anchor map: \(\rho_\mathcal{V} = \rho + \rho^* (\rho : E \rightarrow TM)\). 

\(\mathcal{V}, [\cdot, \cdot]_\mathcal{V}, \rho_\mathcal{V}, (\cdot, \cdot)_\pm\) defines a Vaisman algebroid
The main result of the first half part is the following two points:

[Mori-Sasaki-K.S.,1901.04777]

- The Vaisman algebroid does not require the derivation condition.
- We construct the Vaisman algebroid with a Lie algebroid pair. (not a Lie bialgebroid)

These points were talked by Ms. Mori.
Our Result (of the second part)

- DFT geometry: given by a para-Hermitian manifold $\mathcal{M}$
- using a para-complex structure
  $\Rightarrow$ we split the tangent bundle $T\mathcal{M}$ into 2 distributions $L, \tilde{L}$

Main Result of This Talk

[ Mori-Sasaki-K.S., 1901.04777 ]

- We define a Lie algebroid on $L$ (resp. $\tilde{L}$).
- We construct a Vaisman algebroid by using a Lie algebroid pair $(L, \tilde{L})$ in the DFT setup.
Para-Hermitian Geometry
Doubled Spacetime Geometry

The coordinates in Double Field Theory are \((x^{\mu}, \tilde{x}_{\mu})\).

- \(x^{\mu}\): conjugate to the Kaluza-Klein mode (momentum)
- \(\tilde{x}_{\mu}\): conjugate to the string winding mode

\(\Rightarrow\) Spacetime is doubled!

Doubled spacetime: described by para-Hermitian manifold

[Vaisman '13]

Before introducing a para-Hermitian mfd., we define an almost para-complex mfd.
An almost para-complex manifold is given by \((\mathcal{M}, K)\).

- a differentiable manifold \(\mathcal{M}\)
- the vector bundle endomorphism \(K \in \text{End}(T\mathcal{M})\)
- \(K\) is an almost para-complex structure that satisfies \(K^2 = +1\)
Almost Para-Hermitian Manifold

An almost para-Hermitian manifold is given by \((\mathcal{M}, K, \eta)\):

- an almost para-complex manifold \((\mathcal{M}, K)\)
- a neutral metric \(\eta : T\mathcal{M} \times T\mathcal{M} \rightarrow \mathbb{R}\)
- \(K\) and \(\eta\) satisfy the compatibility condition:
  \[ \eta(K(X), K(Y)) = -\eta(X, Y) \]
Para-Hermitian Manifold

Para-Hermitian manifold \((\mathcal{M}, K, \eta)\):

- the integrability condition is imposed: \(N_K(X, Y) = 0\)

the Nijenhuis tensor is defined by \((X, Y \in T\mathcal{M})\)

\[N_K(X, Y) = \frac{1}{4}\left\{[K(X), K(Y)] + [X, Y] - K([K(X), Y] + [X, K(Y)])\right\}\]

\[
\begin{align*}
\text{almost para-complex} & \quad N_K = 0 \\
\text{para-complex} & \quad \eta \\
\text{para-Hermitian} & \quad N_K = 0 \\
\text{almost para-Hermitian} & \quad \eta
\end{align*}
\]
By using the almost para-complex structure: $K^2 = 1$

\[ T\mathcal{M} = L \oplus \tilde{L} \]

- $L$: eigenbundle associated with the eigenvalue $K = +1$
- $\tilde{L}$: associated with $K = -1$

This decomposition is performed via the projection operators

\[ P = \frac{1}{2}(1 + K), \quad \tilde{P} = \frac{1}{2}(1 - K) \]

$L, \tilde{L}$ are distributions of $T\mathcal{M}$

(a distribution: a generalisation of a vector sub-bundle)
Integrability of Distributions

Since $K$ is the para-complex strc., we can decompose $N_K$:

$$N_K(X, Y) = N_P(X, Y) + N_{\tilde{P}}(X, Y)$$

where

$$N_P(X, Y) = \tilde{P}[P(X), P(Y)], \quad N_{\tilde{P}}(X, Y) = P[\tilde{P}(X), \tilde{P}(Y)]$$

If the tensor $N_P$ ($N_{\tilde{P}}$) vanishes, the distribution $L$ ($\tilde{L}$) is involutive.

**Frobenius Theorem**

A distribution $L$ is Frobenius integrable iff $L$ is involutive

The integrability of distributions is independent of each other. 
(cf. the case of complex mfd.)
Frobenius Theorem (alternative rep.)

A subbundle $E \subset T\mathcal{M}$ is integrable iff it is defined by a foliation of $\mathcal{M}$.

When $L$ and $\tilde{L}$ are integrable, then they have foliation structures:

$$L = T\mathcal{F} \quad \text{and} \quad \tilde{L} = T\tilde{\mathcal{F}}$$

- The foliation $\mathcal{F}$ is given by the union of leaves $\bigcup_p M_p$. 

![Diagram of foliation structures](image-url)
Foliation Structure

**Frobenius Theorem (alternative rep.)**

A subbundle $E \subset T\mathcal{M}$ is integrable iff it is defined by a foliation of $\mathcal{M}$.

When $L$ and $\tilde{L}$ are integrable, then they have foliation structures:

$$L = T\mathcal{F} \quad \text{and} \quad \tilde{L} = T\tilde{\mathcal{F}}$$

- A leaf $M_p$ is a subspace of $\mathcal{F}$ that pass through a point $p \in \mathcal{M}$. 
Foliation Structure

Frobenius Theorem (alternative rep.)

A subbundle $E \subset T\mathcal{M}$ is integrable iff it is defined by a foliation of $\mathcal{M}$.

When $L$ and $\tilde{L}$ are integrable, then they have foliation structures:

$$ L = T\mathcal{F} \quad \text{and} \quad \tilde{L} = T\tilde{\mathcal{F}} $$

- For $\mathcal{F}$, the local coordinate $x^\mu$ is given along a leaf $M_p$. 

![Diagram of foliations](image)
Foliation Structure

**Frobenius Theorem (alternative rep.)**

A subbundle $E \subset T\mathcal{M}$ is integrable iff it is defined by a foliation of $\mathcal{M}$

When $L$ and $\tilde{L}$ are integrable, then they have foliation structures:

$$L = T\mathcal{F} \quad \text{and} \quad \tilde{L} = T\tilde{\mathcal{F}}$$

- The one for the transverse directions to leaves is $\tilde{x}_\mu$. 

[Diagram showing foliation structures $\mathcal{F}$ and $\tilde{\mathcal{F}}$ with leaves and transverse directions]
The metric $\eta$ over $\mathcal{M}$ can be seen as a map

$$\eta : T\mathcal{M} = L \oplus \tilde{L} \to T^*\mathcal{M} = L^* \oplus \tilde{L}^*$$

By using $\eta$, the following isomorphisms are defined

$$\phi^+ : \tilde{L} \to L^* \quad \text{and} \quad \phi^- : L \to \tilde{L}^*$$

Given $\phi^+, \phi^-$, the following new isomorphisms are defined

$$\Phi^+ : T\mathcal{M} \to L \oplus L^* \quad \text{and} \quad \Phi^- : T\mathcal{M} \to \tilde{L} \oplus \tilde{L}^*$$

The map $\Phi^+$ is

- called the natural isomorphism
- utilized to relate DFT and Hitchin’s Generalized Geometry
Doubled Aspects of Vaisman Algebroid in DFT
Let us define a Lie algebroid structure in DFT.

By using the following two theorems,

- Multi-vectors on a manifold define a Gerstenhaber algebra by the Schouten bracket [Tulczyjew '74]
- A Lie algebroid structure over a vector bundle $V \to M$ and a Gerstenhaber algebra over multi-vectors $\Gamma(\wedge^\bullet V)$ are equivalent [Vaintrob '97]

a Lie algebroid is defined by the exterior algebra of multi-vectors in DFT.
Multi-vector on Para-Hermitian Mfd.

We define a natural exterior alg. on the tangent bundle over an almost para-complex mfd. $\mathcal{M}$

- A set of doubled multi-vectors: $\hat{A}^k(\mathcal{M}) = \Gamma(\bigwedge^k T\mathcal{M})$
- We define $\mathcal{A}^{r,s}(\mathcal{M})$ as the section of $(\bigwedge^r L) \wedge (\bigwedge^s \tilde{L})$
Multi-vector on Para-Hermitian Mfd.

We define a natural exterior alg. on the tangent bundle over an almost para-complex mfd. $\mathcal{M}$

- A set of doubled multi-vectors: $\hat{\mathcal{A}}^k(\mathcal{M}) = \Gamma(\wedge^k T\mathcal{M})$

- We define $\mathcal{A}^{r,s}(\mathcal{M})$ as the section of $(\wedge^r L) \wedge (\wedge^s \tilde{L})$

We obtain the decomposition: $\hat{\mathcal{A}}^k(\mathcal{M}) = \bigoplus_{k=r+s} \mathcal{A}^{r,s}(\mathcal{M})$

This decomposition is given by the canonical projection operator

$$\pi^{r,s} : \hat{\mathcal{A}}^{r+s}(\mathcal{M}) \to \mathcal{A}^{r,s}(\mathcal{M})$$

$\pi^{r,s}$ is induced by $P$ and $\tilde{P}$
Para-Dolbeault Operator

Then, we define the exterior derivatives acting on $L$ and $\tilde{L}$

\[ d : \mathcal{A}^{r,s}(\mathcal{M}) \to \mathcal{A}^{r,s+1}(\mathcal{M}) \quad (\wedge^s \tilde{L} \to \wedge^{s+1} \tilde{L}) \]
\[ \tilde{d} : \mathcal{A}^{r,s}(\mathcal{M}) \to \mathcal{A}^{r+1,s}(\mathcal{M}) \quad (\wedge^r L \to \wedge^{r+1} L) \]

$d$ and $\tilde{d}$ have the following properties:

\[ d^2 = 0, \quad \tilde{d}^2 = 0, \quad dd + \tilde{d}\tilde{d} = 0. \]

They are called the para-Dolbeault operators.
We describe the DFT setting as the flat para-Hermitian Geometry.

\[(\mathcal{M}, K, \eta) = \left(\mathcal{M}^{2D}, \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)\]

The tangent space \(T\mathcal{M}\) is spanned by \(\partial_M\) (\(M = 1, \ldots, 2D\)).

Vector fields on \(T\mathcal{M}\) are decomposed by projector \(P, \tilde{P}\).

\[\Xi = \Xi^M \partial_M = A^\mu(x, \tilde{x}) \partial_\mu + \alpha_\mu(x, \tilde{x}) \tilde{\partial}^\mu\]

where \(\Xi \in \Gamma(T\mathcal{M})\), \(A \in \Gamma(L)\), \(\alpha \in \Gamma(\tilde{L})\)

\(x^M = (x^\mu, \tilde{x}_\mu)\) is the induced decomposition of the local coordinate on the base space \(\mathcal{M}\).
Given a $k$-vector $A \in \Gamma(\wedge^k L)$, we introduce the “odd coordinate” $\zeta_\mu := \partial_\mu$

$$A = \frac{1}{k!} A^{\mu_1 \cdots \mu_k} \partial_{\mu_1} \wedge \cdots \wedge \partial_{\mu_k} = \frac{1}{k!} A^{\mu_1 \cdots \mu_k} \zeta_{\mu_1} \cdots \zeta_{\mu_k}$$

$\zeta_\mu$ can be treated as a Grassmann number

$\implies$ differential $\partial/\partial \zeta_\mu$ is defined by the right derivative

By using $\zeta_\mu$ derivative, the Schouten bracket is given by

$$[A, B]_S = \left( \frac{\partial}{\partial \zeta_\mu} A \right) \partial_\mu B - (-1)^{(p-1)(q-1)} \left( \frac{\partial}{\partial \zeta_\mu} B \right) \partial_\mu A$$

The same def. holds for $[\cdot, \cdot]^*_S$ on $\tilde{L}$: replaced $\zeta_\mu = \partial_\mu \leftrightarrow \zeta^{*\mu} = \tilde{\partial}^\mu$
The Para-Dolbeault operator $\tilde{d}$ maps a $k$-vector to a $(k + 1)$-vector.

Using the local coordinate, we find that the action of $\tilde{d}$ on a $k$-vector $X$ is given by

$$\tilde{d}X = \frac{1}{k!} \tilde{\partial}^\mu X_{\nu_1 \cdots \nu_k} \partial_\mu \wedge \partial_{\nu_1} \wedge \cdots \wedge \partial_{\nu_k}.$$ 

In the DFT realization,

- we find that the exterior derivative $\tilde{d}$ on $L$ and the bracket $[\cdot, \cdot]_S^*$ is compatible. (p.app.3)
- the same discussion holds for the operator $d$ on $\tilde{L}$. 


Now, the exterior algebra of multi-vectors in DFT is defined.

We obtain the Lie algebroid \((\wedge \cdot L, [\cdot, \cdot]_S, d)\)
and its dual Lie algebroid \((\wedge \cdot \tilde{L}, [\cdot, \cdot]^*_{S}, \tilde{d})\) in DFT.

The Vaisman Algebroid is constructed by the Lie algebroid pair \((L, \tilde{L})\) without the derivation condition.

Let us examine the violation of the derivation condition.
Examination of the Derivation Condition

The derivation condition of the Lie bialgebroid

\[ d^*[A, B]_S = [d^*A, B]_S + [A, d^*B]_S, \quad A, B \in \Gamma(L) \]

We examine the d.c. in DFT by explicit calculation.

LHS:

\[ \tilde{d}[A, B]_S \]

\[ = \tilde{\partial}^\mu [A, B]_S \partial_\mu \land \partial_\nu \]

\[ = \tilde{\partial}^\mu (A^\rho \partial_\rho B^\nu - B^\rho \partial_\rho A^\nu) \partial_\mu \land \partial_\nu \]

\[ = (\tilde{\partial}^\mu A^\rho \partial_\rho B^\nu + A^\rho \partial_\rho \tilde{\partial}^\mu B^\nu - \tilde{\partial}^\mu B^\rho \partial_\rho A^\nu - B^\rho \partial_\rho \tilde{\partial}^\mu A^\nu) \partial_\mu \land \partial_\nu, \]
Examination of the Derivation Condition

The derivation condition of the Lie bialgebroid

= Compatibility cond. between Lie algebroid and Lie coalgebroid

\[ d_\ast [A, B]_S = [d_\ast A, B]_S + [A, d_\ast B]_S, \quad A, B \in \Gamma(L) \]

We examine the d.c. in DFT by explicit calculation.

LHS: \( \tilde{d}[A, B]_S = (\tilde{\partial}^\mu A^\rho \partial_\rho B^\nu + A^\rho \partial_\rho \tilde{\partial}^\mu B^\nu - \tilde{\partial}^\mu B^\rho \partial_\rho A^\nu - B^\rho \partial_\rho \tilde{\partial}^\mu A^\nu) \partial_\mu \wedge \partial_\nu \)

RHS (1st):

\[ [\tilde{d}A, B]_S = \left( \frac{\partial}{\partial \zeta_\rho} \tilde{d}A \right) \partial_\rho B - (-1)^0 \left( \frac{\partial}{\partial \zeta_\rho} B \right) \partial_\rho \tilde{d}A \]

\[ = (\tilde{\partial}^\mu A^\rho \zeta_\mu - \tilde{\partial}^\rho A^\mu \zeta_\mu) \partial_\rho B^\nu \zeta_\nu - B^\rho \partial_\rho \tilde{\partial}^\mu A^\nu \zeta_\mu \zeta_\nu \]

\[ = (\tilde{\partial}^\mu A^\rho \partial_\rho B^\nu - \tilde{\partial}^\rho A^\mu \partial_\rho B^\nu - B^\rho \partial_\rho \tilde{\partial}^\mu A^\nu) \partial_\mu \wedge \partial_\nu, \]
The derivation condition of the Lie bialgebroid

\[ d_\ast [A, B]_S = [d_\ast A, B]_S + [A, d_\ast B]_S, \quad A, B \in \Gamma(L) \]

We examine the d.c. in DFT by explicit calculation.

LHS: \( \tilde{d}[A, B]_S = (\tilde{\partial}^\mu A^\rho \partial_\rho B^\nu + A^\rho \partial_\rho \tilde{\partial}^\mu B^\nu - \tilde{\partial}^\mu B^\rho \partial_\rho A^\nu - B^\rho \partial_\rho \tilde{\partial}^\mu A^\nu) \partial_\mu \wedge \partial_\nu \)

RHS (1st): \( [\tilde{d}A, B]_S = (\tilde{\partial}^\mu A^\rho \partial_\rho B^\nu - \tilde{\partial}^\rho A^\mu \partial_\rho B^\nu - B^\rho \partial_\rho \tilde{\partial}^\mu A^\nu) \partial_\mu \wedge \partial_\nu \)

RHS (2nd):

\[ [A, \tilde{d}B]_S = -[\tilde{d}B, A]_S \]

\[ = - (\tilde{\partial}^\mu B^\rho \partial_\rho A^\nu - \tilde{\partial}^\rho B^\mu \partial_\rho A^\nu - A^\rho \partial_\rho \tilde{\partial}^\mu B^\nu) \partial_\mu \wedge \partial_\nu. \]
Examination of the Derivation Condition

The derivation condition of the Lie bialgebroid

\[ = \text{Compatibility cond. between Lie algebroid and Lie coalgebroid} \]

\[ d\ast [A, B]_S = [d\ast A, B]_S + [A, d\ast B]_S, \quad A, B \in \Gamma(L) \]

We examine the d.c. in DFT by explicit calculation.

LHS: \( \tilde{d}[A, B]_S = (\tilde{\partial}^\mu A^\rho \partial_\rho B^\nu + A^\rho \partial_\rho \tilde{\partial}^\mu B^\nu - \tilde{\partial}^\mu B^\rho \partial_\rho A^\nu - B^\rho \partial_\rho \tilde{\partial}^\mu A^\nu)\partial_\mu \wedge \partial_\nu \)

RHS (1st): \( [\tilde{d}A, B]_S = (\tilde{\partial}^\mu A^\rho \partial_\rho B^\nu - \tilde{\partial}^\rho A^\mu \partial_\rho B^\nu - B^\rho \partial_\rho \tilde{\partial}^\mu A^\nu)\partial_\mu \wedge \partial_\nu \)

RHS (2nd): \( [A, \tilde{d}B]_S = -(\tilde{\partial}^\mu B^\rho \partial_\rho A^\nu - \tilde{\partial}^\rho B^\mu \partial_\rho A^\nu - A^\rho \partial_\rho \tilde{\partial}^\mu B^\nu)\partial_\mu \wedge \partial_\nu \)

By using \( \eta^{MN} \), the remaining terms are rewritten as

\[ \tilde{\partial}^\rho A^\mu \partial_\rho B^\nu + \tilde{\partial}^\rho B^\nu \partial_\rho A^\mu = \eta^{MN} \partial_M A^\mu \partial_N B^\nu \]
The derivation condition of the Lie bialgebroid

\[ d^*[A, B]_S = [d^*A, B]_S + [A, d^*B]_S, \quad A, B \in \Gamma(L) \]

We examine the d.c. in DFT by explicit calculation.

We obtain

\[ \tilde{d}[A, B]_S = [\tilde{d}A, B]_S + [A, \tilde{d}B]_S + (\eta^{MN} \partial_M A^\mu \partial_N B^\nu) \partial_\mu \wedge \partial_\nu \]

The last contribution represents the violation of the d.c.

(Recall: the section condition is \( \eta^{MN} \partial_M \ast \partial_N \ast = 0 \))
A double of a pair of Lie algebroids

The derivation condition is violated

\[ \Rightarrow (L, \tilde{L}) \text{ is NOT a Lie bialgebroid} \]
A double of a pair of Lie algebroids

The derivation condition is violated

\[ (L, \tilde{L}) \text{ is NOT a Lie bialgebroid} \]

We define the two structures required for a double of \((L, \tilde{L})\).

- The bilinear form is given by

\[
(A + \alpha, B + \beta) = \frac{1}{2} \left\{ \langle \alpha, B \rangle + \langle \beta, A \rangle \right\}.
\]

- The skew-symmetric bracket (called the Vaisman bracket)

\[
[e_1, e_2]_V = [X_1, X_2]_E + \mathcal{L}_{X_1} \xi_2 - \mathcal{L}_{X_2} \xi_1 + \text{d}(e_1, e_2) -
+ [\xi_1, \xi_2]_E^* + \mathcal{L}_{\xi_1} X_2 - \mathcal{L}_{\xi_2} X_1 - \text{d}^*(e_1, e_2) -.
\]

- The anchor map \(\rho_V = \rho_L + \rho_{\tilde{L}}\), the differential op. \(\mathcal{D} = \text{d} + \tilde{\text{d}}\).

\[ (L \oplus \tilde{L}, [\cdot, \cdot]_V, \rho_V, (\cdot, \cdot)) \text{ defines a Vaisman algebroid.} \]
Discussion on DFT Gauge Symmetry
Gauge Symmetry of DFT

- C-bracket = T-duality covariantised Lie bracket-like structure
  - accommodates the diffeo. and $B$-field gauge symmetry algebra
- The structure of the C-bracket in DFT naturally arises as a Vaisman bracket on a para-Hermitian geometry.
  [Vaisman ’13, Svoboda ’18]

- The Vaisman bracket governs the “extended” gauge symmetry in DFT.
  → what is the “extended” gauge symmetry?
Non-Abelian $B$-field Gauge Symmetry

- Vectors in $\tilde{L}$ are identified with 1-forms in $L^*$ by $\Phi^+$
  Recall: the natural isomorphism $\Phi^+ : L \oplus \tilde{L} \rightarrow L \oplus L^*$
  $\rightarrow [\cdot, \cdot]_{\tilde{L}}$ in $\tilde{L}$ represents the $B$-field gauge algebra

- In string theory, the $B$-field gauge symmetry is Abelian
- However, $[\cdot, \cdot]_{\tilde{L}}$ in the Vaisman bracket is generally non-zero
  $\rightarrow$ in DFT, $B$ gauge sym. should be effectively non-Abelian

$\Rightarrow$ This is the “extended” gauge symmetry.
The section condition is needed

- The geometric realization of the Vaisman algebroid → not necessarily require the section condition

- imposing the S.C., gauge algebra is closed by the C-bracket

- in order that algebra given by C-bracket generates a symmetry, S.C. is necessarily satisfied, either implicit or explicitly
A trivial solution of the section condition

• trivial solution of S.C. is winding derivative vanishing

\[
\tilde{d}f = 0.
\]

giving para-holomorphic condition:

• \([\cdot, \cdot]_{\tilde{L}}\) vanish and C-bracket is reduced to original Courant br.

\[
[X_1, X_2]_C = [X_1, X_2]_L + \mathcal{L}_{X_1} \xi_2 - \mathcal{L}_{X_2} \xi_1 - \frac{1}{2} d(\iota_{X_1} \xi_2 - \iota_{X_2} \xi_1)
\]

\[
+ [\xi_1, \xi_2]_{\tilde{L}} + \tilde{\mathcal{L}}_{\xi_1} X_2 - \tilde{\mathcal{L}}_{\xi_2} X_1 + \frac{1}{2} \tilde{d}(\iota_{X_1} \xi_2 - \iota_{X_2} \xi_1)
\]

• imposing the S.C., non-Abelian \(B\)-field gauge symmetry becomes Abelian \([\cdot, \cdot]_{\tilde{L}} = 0\)
Summary
• $T\mathcal{M}$ is decomposed in $L \oplus \tilde{L}$ by the para-complex struc. $K$.
• We define a Lie algd. struc. through the exterior alg. on $L, \tilde{L}$.
• We provide a geometric realization of the Vaisman algebroid.

• The failure of the d.c. is resolved by imposing the S.C.
• We found an algebraic origin of the S.C.

• The gauge transf. of $B$ should be effectively non-Abelian
Future Direction

- In general setups, there is no need to impose S.C. → Vaisman algebroids would play important roles in applications of DFT

- We expect that similar discussions are applied to the exceptional geometries (exceptional field theories)

- Finite gauge transf. in DFT is governed by an “integrated” versions of Vaisman and Courant algebroids → groupoid(-like) structures? (cf. Lie’s 3rd theorem)
Thank you for your attention!
ご清聴ありがとうございます。
Backup
The Section Condition: needed to make DFT be a physical theory

- the physical spacetime: a $D$-dimensional space (not $2D$)
- S.C. is a constraint that reduces extra d.o.f.
- S.C. originally came from the LMC of closed strings
- S.C. is necessary due to
  - the closedness of the DFT gauge algebra
    $$[\hat{\mathcal{L}}_{\Xi_1}, \hat{\mathcal{L}}_{\Xi_2}] \approx \hat{\mathcal{L}}_{[\Xi_1,\Xi_2]} \mod \text{S.C.}$$
  - the gauge invariance of the DFT action
    $$\delta_{\Xi} S_{\text{DFT}} = \hat{\mathcal{L}}_{\Xi} S_{\text{DFT}} \approx 0 \mod \text{S.C.}$$
Example: the $r = 2, s = 0$ case

The projectors in a para-complex mfd. w/ $K = \text{diag}(-1, +1)$:

\[
P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

The component expression of $T \in \hat{A}^2$: $T^{MN} = \begin{pmatrix} t_{\mu\nu} & t_{\mu}^{\nu} \\ t^{\mu}_{\ \nu} & t^{\mu\nu} \end{pmatrix}$.

The canonical projector $\pi^{2,0}$ defined through $P$ is given by

\[
P^{M}_{\ K} T^{KL} P^{N}_{\ L} = \begin{pmatrix} 0 & 0 \\ 0 & t^{\mu\nu} \end{pmatrix} \quad (t^{\mu\nu} \in A^{2,0}).
\]

This implies $\pi^{2,0}(T^{MN}) = t^{\mu\nu}$.

The other projectors $\pi^{1,1}, \pi^{0,2}$ are defined similarly.
Compatibility between $\tilde{d}$ and $[\cdot, \cdot]_S^*$

The general relation of the para-Dolbeault operator:

$$\tilde{d}A(\alpha_1, \ldots \alpha_k) = \sum_{i=1}^{k+1} (-1)^{i+1} \rho_{\tilde{L}}(\alpha_i) \cdot (A(\alpha_1, \ldots, \tilde{\alpha}_i, \ldots, \alpha_k))$$

$$+ \sum_{i<j} (-1)^{i+j} A([\alpha_i, \alpha_j]_S^*, \alpha_1, \ldots, \tilde{\alpha}_i, \ldots, \tilde{\alpha}_j, \ldots, \alpha_k).$$

For example the case of $k = 1$, we have

$$\tilde{d}A(\alpha_1, \alpha_2) = \rho_{\tilde{L}}(\alpha_1) \cdot (A(\alpha_2)) - \rho_{\tilde{L}}(\alpha_2) \cdot (A(\alpha_1)) - A([\alpha_1, \alpha_2]_S^*).$$

Then in the DFT realization, since $\rho_{\tilde{L}}(\alpha_1) = \alpha_1^\mu \tilde{\partial}^\mu$, we have

$$A([\alpha_1, \alpha_2]_S^*)$$

$$= \alpha_1^\mu \tilde{\partial}^\mu (A^\nu \alpha_2^\nu) - \alpha_2^\nu \tilde{\partial}^\nu (A^\mu \alpha_1^\mu) - (\tilde{\partial}^\mu A^\nu - \tilde{\partial}^\nu A^\mu) \alpha_1^\mu \alpha_2^\nu$$

$$= A^\mu (\alpha_1^\nu \tilde{\partial}^\nu \alpha_2^\mu - \alpha_2^\nu \tilde{\partial}^\nu \alpha_1^\mu).$$
• For a vector bundle $E \to M$, a \textit{distribution} $D$ in $E$ assigns to each $x \in M$ a vector subspace $D_x \subseteq E_x$.

• The dimension of $D_x$ is called the \textit{rank} of $D$ at $x$.

• If the rank $D_x$ is independent of $x$, the distribution is called \textit{regular}.

• If there is a smooth local section $v$ of $E$ (s.t. $v(y) \in D_y$ and $v(x) = v_0$) for any $x \in M$ and $v_0 \in D_x$, the distribution is called \textit{smooth}.

• A smooth and regular distribution is a \textit{subbundle}.
• GA is a $\mathbb{Z}$ graded-commutative algebra with a Lie bracket of degree $-1$ satisfying the Poisson identity
• the degree of an element $a$ is denoted by $|a|$
• satisfying the following properties
  • (the product (degree 0)): $|ab| = |a| + |b|$
  • (Lie bracket (degree -1)): $|[a, b]| = |a| + |b| - 1$
  • (associativity): $(ab)c = a(bc)$
  • (graded-commutativity): $ab = (-1)^{|a||b|}ba$
  • (Poisson identity): $[a, bc] = [a, b]c + (-1)^{(|a|-1)|b|}b[a, c]$
  • (skew-symmetry): $[a, b] = -(1)^{(|a|-1)(|b|-1)}[b, a]$
  • (Jacobi identity):
    \[ [a, [b, c]] = [[a, b], c] + (-1)^{(|a|-1)(|b|-1)}[b, [a, c]] \]
• GA differ from Poisson superalgebras in that the Lie bracket has degree $-1$ rather than degree $0$
A Gerstenhaber algebra is a Poisson 2-algebra

A Poisson 2-algebra is a Poisson algebra in graded vector spaces with Poisson bracket of degree $-1$

The exterior algebra of a Lie algebra is a Gerstenhaber algebra

The differential forms on a Poisson manifold form a Gerstenhaber algebra

The multivector fields on a manifold form a Gerstenhaber algebra using the Schouten-Nijenhuis bracket

A Batalin-Vilkovisky algebra has an underlying Gerstenhaber algebra