Four lectures of solitons in Classical Field Theory

Ya Shnir
Modified KdV Equation: Compactons

**KdV equation:**

\[ u_t + 6uu_x + u_{xxx} = 0 \]

\[ \rightarrow K(2,1) \]

**Sequence of modified KdV equations**

\[ K(m,n): \quad u_t + (u^m)_x + (u^n)_{xxx} = 0; \quad m > 0, \quad 1 < n \leq 3 \]

\[ \rightarrow K(2,2): \quad u_t + (u^2)_x + (u^2)_{xxx} = 0; \quad \text{Note: the dispersion is non-linear} \]

\[ u = \begin{cases} \frac{4v}{3} \cos^2[(x - vt)/4]; & |x - vt| \leq 2\pi \\ 0 & |x - vt| \gg 2\pi \end{cases} \]

**Homework:** Prove it!

**Compacton** - soliton solution with a compact support
**Modified KdV Equation: Compactons**

**KdV equation:**

\[ u_t + 6u u_x + u_{xxx} = 0 \]

\[ K(2,1) \]

**Sequence of modified KdV equations**

\[ K(m,n): \quad u_t + (u^m)_x + (u^n)_{xxx} = 0; \quad m > 0, \quad 1 < n \leq 3 \]

\[ K(3,3): \quad u_t + (u^3)_x + (u^3)_{xxx} = 0; \quad \text{Note: the dispersion is non-linear} \]

\[ u = \begin{cases} \sqrt{\frac{3v}{2}} \cos \frac{x-\nu t}{3}; & |x - \nu t| \leq 3\pi/2 \\ 0 & |x - \nu t| \gg 3\pi/2 \end{cases} \]

**Integrals of motion:**

- **Mass + Momentum**
  
  \[ I_1 = u; \quad I_2 = u^3 \]

- **2 Continuos families**
  
  \[ I_3 = u \sin x; \quad I_4 = u \cos x \]
Nonlinear Schrödinger Equation

Non-linear Schrödinger equation:

\[ i\psi_t + \psi_{xx} + 2\sigma|\psi|^2\psi = 0 \]

\[ \sigma = \pm 1 \]

Lagrangian:

\[ L = i(\psi\psi^* - \psi_t\psi^*) + \frac{1}{2}|\psi_x|^2 - \frac{\sigma}{2}|\psi|^4 \]

Symmetries:

- **Translational invariance:** \( t \to t, \quad x \to x + \delta x, \quad \psi \to \psi \)
- **Time invariance:** \( t \to t + \delta t, \quad x \to x, \quad \psi \to \psi \)
- **Scale invariance:** \( t \to \alpha^2 t, \quad x \to \alpha x, \quad \psi \to \psi/\alpha \)
- **Galilean invariance:** \( t \to t, \quad x \to x - ct, \quad \psi \to \psi e^{ic(x - \frac{c}{2}t)/2} \)

Hamiltonian:

\[ H = \frac{1}{2}|\psi_x|^2 + \frac{\sigma}{2}|\psi|^4 \]

One-parameter Lie group of symmetries
**NSE Solitons**

\[ i\psi_t + \psi_{xx} + 2\sigma|\psi|^2\psi = 0 \]

Ansatz for the soliton solution: \( \psi(x, t) = u(x)e^{i\phi(t)} \)

\[ \frac{d\phi}{dt} = \frac{u_{xx}}{u} + 2\sigma u^2 = C = \text{const} \quad \Rightarrow \quad \phi = Ct \]

\[ u_{xx} = -2\sigma u^3 + Cu \quad \text{integrating factor} \ u_x \quad \Rightarrow \quad (u_x)^2 = -\sigma u^4 + Cu^2 + C_0 \]

Shape of the solitary waves depends on the sign of \( \sigma \)

**Bright Soliton:** \( \sigma = 1 \) (Focusing NLS)

\[ (u_x)^2 = -u^4 + Cu^2 + C_0 \]

Boundary conditions: \( u = u' = 0, \) as \( x \to \pm \infty \)

Simplest solution (\( C = 1 \)):

\( u = \text{sech} \ x; \quad \phi = t \quad \Rightarrow \quad \psi = \text{sech} \ x e^{it} \)

Using Galilean and scale symmetry:

\[ \psi = A \text{ sech} \ A(x - ct)e^{i\left(\frac{\sigma}{2}x + (A^2 - \frac{c^2}{4})t\right)} \]

**Homework:** Consider \( C = -1 \)

Two-parameter family of bright solitons
**Instability of the bright soliton:** for sufficiently large values of $c$ the envelope has spatial oscillations of the same period as the carrier wave.

**Dark Soliton:** $\sigma = -1$ (Defocusing NLS)

\[(u_x)^2 = u^4 + Cu^2 + C_0\]

Simplest solution ($C=-1$, $C_0=\frac{1}{4}$):

\[
\psi = \frac{1}{\sqrt{2}} \tanh \frac{x}{\sqrt{2}} e^{-it}
\]

Homework: Consider $C=1$

Two-parameter family of dark solitons

Using Galilean and scale symmetry:

\[
\psi = \frac{A}{\sqrt{2}} \tanh \frac{A(x - ct)}{\sqrt{2}} e^{i\left(\frac{c}{2} x (A^2 + \frac{c^2}{4}) t\right)}
\]
Focusing NSE: Breathers

Freak (rogue) wave: a single wave or a very short wave group with a significantly larger steepness than the surrounding waves – Breather solution of the NLS equation

\[
\psi = \frac{\cos(\Omega t - 2ik) - \cosh(k) \cosh(px)}{\cos(\Omega t) - \cosh(k) \cosh(px)} e^{2it}, \quad \Omega = 2 \sinh(2k), \quad p = 2 \sinh k
\]

Note: While for a bright soliton there is always a reference frame where the envelope \(|\psi|\) is stationary, this is not so for breathers ("dynamical solitons")

Limit of zero breathing period \(k \to 0\):

**Peregrine breather (1983)**

\[
\psi = \left[1 - \frac{4(1 + 4it)}{1 + 4x^2 + 16t^2}\right] e^{2it}
\]
Boussinesq equation

Recall: The Lax pair for the KdV equation:

\[
\mathbf{L}\psi \equiv (-\partial_x^2 - u)\psi = \lambda \psi \quad \psi_t = \mathbf{A}\psi \equiv (-4\partial_x^3 - 6u\partial_x - 3u_x)\psi
\]

Another example: The Lax pair for the Boussinesq-type equation

\[
\mathbf{L}\psi \equiv (-\partial_{xxx} + u\partial_x + v)\psi = \lambda \psi \quad \psi_t = \mathbf{A}\psi \equiv (\partial_{xx} + \frac{2}{3}u)\psi
\]

\[
\frac{d}{dt}(-\partial^3 + u\partial + v) = [\mathbf{A}, \mathbf{L}] = (2v' - u'')\partial + v'' - \frac{2}{3}u''' - \frac{2}{3}uu'
\]

\[
\begin{aligned}
\dot{u} &= 2v' - u''; \\
\dot{v} &= v'' - \frac{2}{3}u''' - \frac{2}{3}uu'
\end{aligned}
\]

Boussinesq-type equations describe waves which can propagate both to the right, and to the left ("the two-way long-wave equations").

\[
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - u_{xx} + 3(u^2)_{xx} + u_{xxxx} &= 0
\end{aligned}
\]

Travelling wave solution has the form \( u \equiv u(\theta) \) where \( \theta = x - vt \)

\[
\begin{aligned}
u(x, t) &= 2a^2 \text{sech}^2(a(x - vt)); \\
v &= \pm \sqrt{1 - 4a^2}
\end{aligned}
\]
**Fermi-Pasta-Ulam system**

*E Fermi, J Pasta, and S Ulam (1955):* numerical study of the dynamics of an anharmonic chain of particles connected to their nearest neighbours by weakly nonlinear springs

**MANIAC-1**

*(Mathematical Analyzer Numerical Integrator And Computer)*

\[ m\ddot{u}_n = f(u_{n+1} - u_n) - f(u_{n} - u_{n-1}); \]

\[ u_n \] displacement of the \( n \)-th particle from the equilibrium

\[ m\ddot{u}_n = k(u_{n+1} - 2u_n + u_{n-1}) + \alpha \left[ (u_{n+1} - u_n)^2 - (u_n - u_{n-1})^2 \right] \]

Weak non-linearity

A general solution of the linearized system (\( \alpha = 0 \)) is given by the expansion in the normal modes:

\[ u_n^k(t) = A_k \sin \left( \frac{k\pi n a}{N + 1} \right) \cos(\omega_k t + \delta_k) \]

\[ \omega_k = 2\sqrt{\frac{k}{m}} \sin \left( \frac{k\pi a}{2(N + 1)} \right) \]
STUDIES OF NON LINEAR PROBLEMS

E. FERMI, J. PASTA, and S. ULAM

A one-dimensional dynamical system of 64 particles with forces between neighbors containing nonlinear terms has been studied on the Los Alamos computer MANIAC I. The nonlinear terms considered are quadratic, cubic, and broken linear types. The results are analyzed into Fourier components and plotted as a function of time.

The results show very little, if any, tendency toward equipartition of energy among the degrees of freedom.

\[
x_i'' = (x_{i+1} + x_{i-1} - 2x_i) + \alpha \left[ (x_{i+1} - x_i)^2 - (x_i - x_{i-1})^2 \right] \\
(i = 1, 2, \cdots, 64),
\]
All numerical simulations of the Fermi-Pasta-Ulam problem were performed by Mary Tsingou.

**Note:** There is no energy transfer between the modes in the linear approximation. In the nonlinear chain ($\alpha \neq 0$) modes become coupled. It was expected that if all the initial energy was put into a few lowest modes, the nonlinear coupling would yield equal distribution of the energy among the normal modes.

**However:** If the energy was initially in the mode of lowest frequency, it returned almost entirely to that mode after interaction with a few other low frequency modes.
From Fermi-Pasta-Ulam to Boussinesq equation

- **Continuum approximation** *(Zabusky and Kruskal (1965))*:
  \[ u_n(t) = u(x_n, t) = u(na, t); \quad u_{n\pm 1}(t) = u(x_n \pm a, t) \]

- **Gradient expansion**:
  \[ u_{n\pm 1}(t) \approx u(x_n, t) \pm au'(x_n, t) + \frac{a^2}{2} u''(x_n, t) \pm \frac{a^3}{3!} u'''(x_n, t) + \frac{a^4}{4!} u''''(x_n, t) + \ldots \]

- **FPU**:
  \[ m\ddot{u}_n = k(u_{n+1} - 2u_n + u_{n-1}) + \alpha \left[ (u_{n+1} - u_n)^2 - (u_n - u_{n-1})^2 \right] \]

- **Boussinesq**:
  \[ u_{tt} - c^2 u_{xx} = \varepsilon c^2 (u_x u_{xx} + \delta^2 u_{xxxx}) \]
  \[ c^2 = \frac{ka^2}{m}; \quad \varepsilon = \frac{2\alpha a}{k}; \quad \delta^2 = \frac{a^2}{12\varepsilon} \]

**Note:** the leading order nonlinear and dispersive contributions in the r.h.s. are balanced at the same order of \( \varepsilon \)

J. Boussinesq originally derived a system of two first-order (in time) equations for weakly nonlinear surface waves in shallow water (1881)

\[ \begin{cases} \eta_t + u_x + (\eta u)_x = 0 \\ u_t + \eta_x + uu_x - u_{xx}t = 0 \end{cases} \]

\( \eta \) is the free surface elevation

\( u \) is the horizontal velocity
From Boussinesq equation to KdV

\[ u_{tt} - c^2 u_{xx} = \varepsilon c^2 (u_x u_{xx} + \delta^2 u_{xxxx}) \]

- **Asymptotic multiple-scale expansion:**  
  \[ u(x, t) = f(\theta, T) + \varepsilon v(x, t) \]
  \[ \theta = x - ct, \quad T = \varepsilon t \]

  \[ v_{tt} - c^2 v_{xx} = 2c f_T \theta + c^2 f_{\theta\theta} + c^2 \delta^2 f_{\theta\theta\theta\theta} + \ldots \]

**Note:** The function \( v(x, t) \) grows linearly in \( \bar{\theta} = x + ct \) unless the r.h.s is not zero:

\[ 2c f_T \theta + c^2 f_{\theta\theta} + c^2 \delta^2 f_{\theta\theta\theta\theta} = 0 \]

- **Change of variables:**  
  \[ q = \frac{f_\theta}{6}; \quad \tau = \frac{cT}{2} \]

In 1965 Kruskal and Zabusky numerically studied the dynamics of the KdV equation with sinusoidal initial conditions (for small \( \delta = 0.022 \) with periodic boundary conditions)

- The appearing solitary waves interact with each other elastically
- They have called the waves **solitons** since they behave like particles
- Explanation of the FPU recurrence as property of the system of solitons moving with different speed. Since the system studied was of finite length, solitons eventually reassembled in the \((x, t)\) plane and approximately recreated the initial configuration
The original FK model (1938) was proposed to describe dislocations in metals. The atoms are treated as a one-dimensional chain subjected to an external periodic potential produced by the surrounding atoms.

\[
L = T - V = \sum_n \frac{m u_n^2}{2} - \frac{\alpha}{2} \sum_n (u_{n+1} - u_n)^2 - V_0 \sum_n \left(1 - \cos \frac{2\pi u_n}{a}\right)
\]

**Rescaling I:**
\[u_n \rightarrow \frac{2\pi u_n}{a}; \quad t \rightarrow \frac{2\pi}{a} \sqrt{\frac{V_0}{m}} t; \quad \alpha \rightarrow \alpha \left(\frac{a}{2\pi}\right)^2 \frac{V_0}{\alpha}
\]

\[\ddot{u}_n - \alpha (u_{n+1} - 2u_n + u_{n-1}) = -\sin u_n \times e^{ikn}
\]

\[V_0 = 0
\]

\[
\sum_{n=-\infty}^{\infty} u_n e^{-ikn} = 2\alpha (\cos k - 1)
\]

Uncoupled oscillators:
\[u(k, t) = A(k) \cos \omega t + B(k) \sin \omega t; \quad \omega^2 = 2\alpha (1 - \cos k)
\]
\[ T = \frac{I}{2} \sum_n \left( \frac{\partial \phi(x_n, t)}{\partial t} \right)^2 \]

\[ U = \frac{\alpha}{2} \sum_n [\phi(x_{n+1}, t) - \phi(x_n, t)]^2 + \sum_n V[x_n, t] \]

\[ V[x_n] = -mgl \left( 1 - \cos \phi(x_n, t) \right) \]

\[ \ddot{\phi}_n - \alpha(\phi_{n+1} - 2\phi_n + \phi_{n-1}) = -\sin \phi_n \]

\[ \phi(x_{n\pm 1}) \approx \phi(x_n) \pm a \frac{\partial u(x_n)}{\partial x} + \frac{a^2}{2} \frac{\partial^2 \phi(x_n)}{\partial x^2} + \frac{a^3}{3!} \frac{\partial^3 \phi(x_n)}{\partial x^3} + \frac{a^4}{4!} \frac{\partial^4 \phi(x_n)}{\partial x^4} + \ldots \]

\[ \phi(x_{n+1}) - 2\phi(x_n) + \phi(x_{n-1}) \approx a^2 \frac{\partial^2 \phi(x_n)}{\partial x^2} \]

\[ \phi_{xx} - \alpha \phi_{xx} + mgl \sin \phi = 0 \]

\[ x \to \frac{x}{a \sqrt{\frac{mgl}{\alpha}}}; \quad t \to t \sqrt{mgl} \quad \Rightarrow \quad u_{tt} - u_{xx} + \sin u = 0 \]

This is the sine-Gordon equation
Sine-Gordon model: scalar field theory

\[ L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi); \quad U(\phi) = 1 - \cos \phi \]

**Note:** The model is Lorentz-invariant

Field equation:

\[ \phi_{tt} - \phi_{xx} + \sin \phi = 0 \quad \phi \rightarrow \phi \pm 2\pi n, \quad n \in \mathbb{Z} \]

Non-trivial static solutions: the function \( \phi(x) \) interpolates between \( \phi(-\infty) = 0 \) and \( \phi(+\infty) = 2\pi \)

**Integration:** \( \phi_{xx} = \sin \phi \quad \Rightarrow \quad \int \phi_{xx} dx = \int \sin \phi \, dx \)

Boundary condition: \( \phi_x \rightarrow 0 \) as \( x \rightarrow \pm \infty \) \( \Rightarrow \quad C' = 1 \)

It looks like an equation of motion of a “particle” in the effective potential

**Separating the variables:**

\[ x - x_0 = \pm \int \frac{d\phi}{\sqrt{2(1 - \cos \phi)}} = \pm \int \frac{d\phi}{2 \sin(\phi/2)} = \int d(\ln \tan \frac{\phi}{4}) \]

**Kink solution:**

\[ \phi = \pm 4 \arctan \exp (x - x_0) \]

Busted kink:

\[ x \rightarrow \frac{x - vt}{\sqrt{1 - v^2}} \]
Sine-Gordon model: scalar field theory

**Note:**
- Unlike KdV solitary wave the function $\phi(x)$ does not go to 0 as $x \to \pm \infty$
- The amplitude of the SG soliton is independent of its velocity
- SG soliton is **topological**
- The SG model is integrable
- The SG model is relativistic-invariant

For small $\phi(x)$ \[ 1 - \cos \phi \approx \frac{\phi^2}{2} + \frac{\phi^4}{4} + \ldots \]

**$\phi^4$ model**
\[ L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - (\phi^2 - 1)^2 \]

Kink solution: \[ \phi_{K\bar{K}} = \pm 4 \arctan(e^{-x+x_0}) \]

Energy density: \[ E = \frac{4}{\cosh^2(x - x_0)} \quad \text{Mass of the kink:} \quad M = \int E \, dx = 8 \]

**Topological charge:** \[ Q = \frac{1}{2\pi} \int d\phi \frac{\partial \phi}{\partial x} = \text{-n-fold cover of } [0,2\pi] \]

**Topological current:** \[ J_\mu = \frac{1}{2\pi} \epsilon_{\mu\nu} \partial^\nu \phi, \quad \partial^\mu J_\mu \equiv 0 \]

**Note:** This is not a Noether current!
Topology primer: maps and windings

Kinks in 2d:

Space: $+\infty$ $-\infty$  
Vacuum: $+1$ $-1$  
Maps:  

Topological charge: $Q = \frac{1}{2} \int_{-\infty}^{\infty} dx \frac{\partial \phi}{\partial x} = \phi(\infty) - \phi(-\infty)$

Circles: $S^1 \rightarrow S^1$

Space:  
Vacuum:  
Maps:  

$\phi^\alpha = (\sin \varphi; \cos \varphi)$
Circles: $S^1 \rightarrow S^1$

**Topological charge:**

\[ Q = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \ \varepsilon_{\alpha\beta} \frac{d\phi^\alpha}{d\varphi} \phi^\beta \]

Vacuum:

\[ \phi^\alpha = (0, 1) \]

\[ \phi^\alpha = (\sin \varphi; \cos \varphi) \]

\[ \phi^\alpha = (\sin 2\varphi; \cos 2\varphi) \]
sine-Gordon equation: Light-cone coordinates

**Light cone coordinates:**

\[ x_{\pm} = \frac{1}{2}(x \pm t) \]

\[ \partial_x = \frac{1}{2} \partial_+ + \frac{1}{2} \partial_-; \quad \partial_t = \frac{1}{2} \partial_+ - \frac{1}{2} \partial_- \]

\[ \partial_t^2 - \partial_x^2 = -\partial_- \partial_+ \]

Then the SG equation becomes

\[ \partial_- \partial_+ \phi = \sin \phi \]

**First fundamental form of the 2-dim surface covered by \( x_{\pm} \):**

\[ ds^2 = dx_-^2 + dx_+^2 + 2 \cos \phi dx_- dx_+ \]

\( \phi \) - an angle between the asymptotic lines \( x_- = \text{const}; \quad x_+ = \text{const} \)

**Second fundamental form:** \( \mathbb{I} = \sin \phi \ d x_- d x_+ \)

**Equation on the Gaussian curvature \( K \):**

\[ \partial_- \partial_+ \phi + K \sin \phi = 0 \]

The sine-Gordon equation geometrically represents the compatibility equation between the first and the second fundamental forms: \( K = -1 \)
Pseudospherical surfaces of Gaussian curvature $K = -1$

1-Kink

2-Kinks

Breather
Bäcklund transformation: if we have a solution of an integrable system, even a trivial one, there is a transformation which transforms it into a new non-trivial solution.

**Example I:** Laplace equation in 2d \( \Delta u(x, y) = (\partial_x^2 + \partial_y^2)u = 0 \)

Let us take another equation for a new function \( v(x, y) \):

\( \Delta v(x, y) = (\partial_x^2 + \partial_y^2)v = 0 \)

**Note:** the functions \( u(x, t) \) and \( v(x, t) \) are not independent:

\[ \partial_x u = \partial_y v; \quad \partial_y u = -\partial_x v \]

Indeed \( \partial_x (\partial_x u) = \partial_x (\partial_y v), \quad \partial_y (\partial_y u) = -\partial_y (\partial_x v) \), so sum of these two equations yields the original Laplace equation.

Now we take the trivial solution \( v(x, y) = xy \) and plug it into the Bäcklund transformation:

\[ u_x = x; \quad u_y = -y \quad \text{i.e.} \quad u = \frac{1}{2} (x^2 - y^2) \]
Bäcklund transformation for the sine-Gordon model

**sG equation:** \( \partial_- \partial_+ \phi = \sin \phi \)

**sG Bäcklund transformation**

Consider the pair of equations:

\[
\begin{align*}
\partial_+ \psi &= \partial_+ \phi - 2 \lambda \sin \left( \frac{\phi + \psi}{2} \right), \\
\partial_- \psi &= -\partial_- \phi + \frac{2}{\lambda} \sin \left( \frac{\phi - \psi}{2} \right)
\end{align*}
\]

\[
\partial_- \partial_+ \psi = \partial_- \partial_+ \phi - 2 \cos \left( \frac{\phi + \psi}{2} \right) \sin \left( \frac{\phi - \psi}{2} \right) = \partial_- \partial_+ \phi + \sin \phi - \sin \psi
\]

If \( \partial_- \partial_+ \phi = \sin \phi \), then \( \partial_- \partial_+ \psi = \sin \psi \)

**Start with the trivial vacuum solution:** \( \phi = 0 \)

\[
\begin{align*}
\partial_+ \psi &= -2 \lambda \sin(\psi/2); \\
\partial_- \psi &= -2 \lambda^{-1} \sin(\psi/2)
\end{align*}
\]

**Homework:** Prove it!

\[
\psi = 4 \arctan[\exp(-\lambda x_+ - \frac{x_-}{\lambda})]
\]

**Back to original coordinates:** \( \lambda x_+ + \lambda^{-1} x_- = \pm \frac{x - vt}{\sqrt{1 - v^2}} \)

**1-Kink solution:** \( \phi_{K\bar{K}} = \pm 4 \arctan(e^{\pm \frac{x - vt}{\sqrt{1-v^2}}}) \)
**Bäcklund transformation for the sine-Gordon model**

- **SG two-soliton solution:**
  \[
  \phi_0 \xrightarrow{\lambda_1} \psi_1 \xrightarrow{\lambda_2} \phi_2
  \]
  \[
  \lambda_2 \xrightarrow{\psi_2} \lambda_1 \xrightarrow{\phi_2}
  \]

  Eliminating the derivatives in the SG Bäcklund transformation, we obtain (\(\phi_0=0\))

  \[
  \tan \left( \frac{\phi_2}{4} \right) = \left( \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} \right) \tan \left( \frac{\psi_2 - \psi_1}{4} \right)
  \]

  **Recall:**

  \[
  \psi_{1,2} = 4 \arctan e^{\theta_{1,2}}
  \]

  \[
  \theta_{1,2} = \frac{1}{2} \left( \lambda_i x + \lambda_i^{-1} t + C_i \right)
  \]

  Two one-soliton solutions

- Consider asymptotic: \(\theta_2 \gg 1\)

  \[
  \frac{e^{\theta_1} - e^{\theta_2}}{1 + e^{\theta_1+\theta_2}} \to \frac{e^{\theta_1}/e^{\theta_2} - 1}{e^{-\theta_2} + e^{\theta_1}} \sim -e^{-\theta_1}
  \]

- **The symmetric 2-kink solution**
  (head-on collision, identical velocities):

  \[
  \phi_2 = 4 \arctan \left[ \frac{v \sinh \frac{x}{\sqrt{1-v^2}}}{\cosh \frac{vt}{\sqrt{1-v^2}}} \right]
  \]

  \[
  \lambda_2 = -\frac{1}{\lambda_1}; \quad v = \frac{1 - \lambda_1^2}{1 + \lambda_1^2}, \quad \lambda_1 > 0
  \]

  Topological charge: \(Q = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \frac{\partial \phi_2}{\partial x} = 2\)
sine-Gordon model: 2-soliton interactions

**KK-collision**

\( \nu = 0.8 \)

\[ \phi_2 = 4 \arctan \left[ v \sinh \frac{x}{\sqrt{1-v^2}} \cosh \frac{vt}{\sqrt{1-v^2}} \right] \]

**K\bar{K}\text{-collision}**

\( \nu = 0.8 \)

\[ \lambda_2 = \frac{1}{\lambda_1}; \quad v = \frac{1 - \lambda_1^2}{1 + \lambda_1^2}, \quad \lambda_1 > 0 \]

\[ \phi_2 = 4 \arctan \left[ \sinh \frac{vt}{\sqrt{1-v^2}} \cosh \frac{x}{\sqrt{1-v^2}} \right] \]

**Breather:**

\[ v = \frac{i \omega}{\sqrt{1 - \omega^2}} \]

\[ \phi_2 = 4 \arctan \left[ \frac{\sqrt{1 - \omega^2}}{\omega} \sin \omega t \cosh \sqrt{1 - \omega^2} x \right] \]
The sine-Gordon model: 2-soliton interactions

**KK-collision**

\[ \phi_2 = 4 \arctan \left( \frac{\sinh \gamma vt}{v \cosh \gamma x} \right) ; \quad \gamma = \frac{1}{\sqrt{1 - v^2}} \]

\[ = 4 \tan^{-1} \left[ \frac{e^{\gamma vt - \ln v} - e^{-\gamma vt - \ln v}}{e^{\gamma x} + e^{-\gamma x}} \right] \]

Asymptotic: \( t \to -\infty \)

\[ \phi_2 \approx \phi_K \left[ (x + v \left( t + \frac{\delta t}{2} \right) \gamma \right] + \phi_{\bar{K}} \left[ (x - v \left( t - \frac{\delta t}{2} \right) \gamma \right] \]

Asymptotic: \( t \to +\infty \)

\[ \phi_2 \approx \phi_K \left[ (x + v \left( t - \frac{\delta t}{2} \right) \gamma \right] + \phi_{\bar{K}} \left[ (x - v \left( t + \frac{\delta t}{2} \right) \gamma \right] \]

The phase shift: \[ \delta t = 2 \frac{\ln v}{\gamma v} = 2\sqrt{v^2 - 1} \ln v \]
sine-Gordon model: Lax pair formulation

Recall: Lax pair is given by two linear equations

\[
\begin{align*}
\psi_x &= L \psi; \\
\psi_t &= A \psi
\end{align*}
\]

\[
\psi = \begin{pmatrix}
\psi_{11} & \psi_{12} \\
\psi_{21} & \psi_{22}
\end{pmatrix}
\]

\[
\begin{align*}
\psi_{xt} &= L_t \psi + L \psi_t; \\
\psi_{tx} &= A_x \psi + A \psi_x.
\end{align*}
\]

sine-Gordon:

\[
L = i \lambda \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} + \frac{i}{2} \begin{pmatrix}
0 & u_x \\
u_x & 0
\end{pmatrix} = i \lambda \cdot \sigma_3 + \frac{i}{2} u_x \cdot \sigma_1; \quad \lambda \in \mathbb{C}
\]

\[
A = \cos u \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} + \frac{1}{4i \lambda} \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix} = \cos u \cdot \sigma_3 + \frac{1}{4i \lambda} \cdot \sigma_2
\]

\[
L_t = \frac{iu_{tx}}{2} \cdot \sigma_1; \quad A_x = -\frac{1}{4i \lambda} u_x \sin u \cdot \sigma_3 + \frac{1}{4i \lambda} u_x \cdot \sigma_2
\]

\[
[A, L] = \frac{i}{4 \lambda} \cdot \sigma_2 - \frac{i}{4 \lambda} \cdot \sigma_3 + \frac{i}{2} \sin u \cdot \sigma_1
\]

in 0th order in \( \lambda \)

\[
\frac{iu_{tx}}{2} \cdot \sigma_1 = \frac{i}{2} \sin u \cdot \sigma_1
\]

sine-Gordon equation is recovered!
**Interaction between the solitons**

2-solitons solutions

\[ \phi = \phi_K(x - d) + \phi_k(x + d) - 2\pi \]

**Linear perturbations of the kink:**

\[ \phi = \phi_K(x) + \eta(x, t) \]

\[ \phi_K = 4 \arctan e^x \]

\[ \eta(x, t) = \Re \, \xi(x)e^{i\omega t} \]

\[ E_{int}(d) = E_{KK}(d) - 2M \approx 32e^{-2d} \]

**Homework:** Prove it!
Linearized perturbations of the sG kink

\[
\left( -\frac{d^2}{dx^2} + 1 - \frac{2}{\cosh^2 x} \right) \xi(x) = \omega^2 \xi(x)
\]

\[
\hat{a}^\dagger \hat{a} \xi(x) = \omega^2 \xi(x); \quad \hat{a} = \frac{d}{dx} + \tanh x; \quad \hat{a}^\dagger = -\frac{d}{dx} + \tanh x
\]

Vacuum state: \[ \hat{a} \xi_0 \equiv \left( \frac{d}{dx} + \tanh x \right) \xi(x) = 0 \]

\[ \xi_0(x) = \frac{1}{\cosh x} \]

Zero mode

\[ \phi = \phi_K(x) + C \xi_0(x) = 4 \arctan e^x + \frac{C}{\cosh x} = \phi_K + \frac{C}{2} \frac{d\phi_K}{dx} \approx \phi_K \left( x + \frac{C}{2} \right) \]

Continuum modes: \[ \xi_k(x) = (\tanh x + ik)e^{ikx}; \quad \omega_k = \sqrt{1 + k^2} \]

Note: \[ \hat{a} e^{ikx} \equiv \left( \frac{d}{dx} + \tanh x \right) e^{ikx} = \xi_k(x) \]

\[ \xi(-\infty) = (-1 + ik)e^{ikx}; \quad \xi(\infty) = (1 + ik)e^{ikx+\delta}; \quad e^{i\delta} = \frac{ik + 1}{ik - 1} \]
$\phi^4$ model

$L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi); \quad U(\phi) = \frac{\lambda}{4} (\phi^2 - a^2)^2$

Field equation: $\partial_\mu \partial^\mu \phi + \frac{dU}{d\phi} = 0$

Potential energy: $V = \int_{-\infty}^{\infty} dx \left[ \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 + U(\phi) \right]$

Kinetic energy: $T = \frac{1}{2} \int_{-\infty}^{\infty} dx \left( \frac{\partial \phi}{\partial t} \right)^2$

Vacuum: $\phi_0 = \pm a$ \quad Static configuration: $T=0$

Energy bound: $E = V = \int_{-\infty}^{\infty} dx \left[ \frac{1}{\sqrt{2}} \phi' \pm \sqrt{U(\phi)} \right]^2 + \int_{-\infty}^{\infty} dx \sqrt{2U(\phi)} \phi' \geq 0$
**$\phi^4$ model: Applications**

- Phenomenological theory of second order phase transitions
- A model of the displacive phase transitions
- A model of uniaxial ferroelectrics
- A phenomenological theory of the non-perturbative transition in polyacetylene chain
- Condensed matter physics: solitary waves in shape-memory alloys
- Cosmology: model dynamics of the domain walls.
- Biophysics: soliton excitations in DNA double helices.
- Quantum field theory: a model example to investigate transition between perturbative and non-perturbative sectors of the theory.
- A model of quantum mechanical instanton transitions in double-well potential
**$\phi^4$ model: Kink solutions**

$$U(\phi) = \frac{1}{2}(\phi^2 - 1)^2; \quad V = \frac{1}{2} \int_{-\infty}^{\infty} dx \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + (\phi^2 - 1)^2 \right]$$

**Minimum of the energy:**

$$\frac{\partial \phi}{\partial x} = \pm (1 - \phi^2) \quad \Rightarrow \quad x - x_0 = \pm \int \frac{d\phi}{1 - \phi^2} = \pm \tanh^{-1} \phi$$

**Kink solution:**

$$\phi_K = \tanh(x - x_0); \quad \bar{\phi}_K = -\tanh(x - x_0)$$

**Energy density:**

$$\mathcal{E} = \frac{1}{\cosh^4(x - x_0)}$$

**Mass of the kink:**

$$M = \int \mathcal{E} dx = \frac{4}{3}$$

**Topological charge:**

$$Q = \frac{1}{2} \int_{-\infty}^{\infty} dx \frac{\partial \phi}{\partial x} = \frac{1}{2} [\phi(\infty) - \phi(-\infty)]$$

**Topological current:**

$$J_\mu = \frac{1}{2\pi} \varepsilon_{\mu\nu} \partial^\nu \phi, \quad \partial^\mu J_\mu \equiv 0$$
Interaction between the kinks

Kink-antikink pair (a=1, m = √2):

\[ \phi(x) = 1 + \tanh(x - R) - \tanh(x + R) \]

Far away from the pair (somewhere at x \approx 0)

\[ \tanh(x - R) \approx -1 + 2e^{2(x-R)}; \]
\[ \tanh(x + R) \approx 1 - 2e^{-2(x+R)} \]

Interaction energy:

\[ E_{int} \approx -16e^{-2L}, \quad L = 2R \]

Kinks attracts each other with the force

\[ F = \frac{dE_{int}}{dL} \approx 32e^{-2L} \]

Linear oscillations on the static kink background:

\[ \ddot{\phi} - \phi'' - \lambda(a^2 - \phi^2)\phi = 0 \rightarrow \delta \ddot{\phi} - \delta \phi'' + \left[ 4 - \frac{6}{\cosh^2 x} \right] \delta \phi = 0 \]
Linearized perturbations of the $\phi^4$ kink

$$\left( -\frac{d^2}{dx^2} + 4 - \frac{6}{\cosh^2 x} \right) \xi = \omega^2 \xi$$

Reflectionless potential, again!

$$\hat{a} \hat{a}^\dagger \xi(x) = \omega^2 \xi(x); \quad \hat{a} = \frac{d}{dx} + n \tanh x; \quad \hat{a}^\dagger = -\frac{d}{dx} + n \tanh x$$

$$[\hat{a}^\dagger, \hat{a}] = \frac{2n}{\cosh^2 x}$$

Vacuum state:

$$\hat{a} \xi_0 \equiv \left( \frac{d}{dx} + n \tanh x \right) \xi_0^{(n)}(x) = 0 \quad \xi_0^{(n)}(x) = \frac{1}{\cosh^n x}$$

Zero mode

Internal mode:

$$\hat{a}^\dagger \xi_0 = \xi_1 = \frac{\sinh x}{\cosh^2 x} \quad \omega_1 = \sqrt{3}$$

Continuum:

$$\xi_k = e^{ikx} \left( 3 \tanh^2 x - 3ik \tanh x - 1 - k^2 \right)$$

Homework: Prove it!

$n=2$
Linearized perturbations of the $\phi^4$ kink

$$\left(-\frac{d^2}{dx^2} + 4 - \frac{6}{\cosh^2 x}\right)\xi = \omega^2 \xi$$

$$\xi^{(n)}(x) = \frac{1}{\cosh^n x}; \quad \omega_0 = 0$$

Zero mode:

$$\hat{a}^\dagger \xi_0 = \xi_1 = \frac{\sinh x}{\cosh^2 x}; \quad \omega_1 = \sqrt{3}$$
\[ \omega_k^2 = (4 + k^2); \quad \xi_k(x) = \Re \left[ e^{ikx} (3 \tanh^2 x - 3ik \tanh x - 1 - k^2) \right] \]

Coupling \( \int dx \eta_k \eta_0 \) (negative radiation pressure)

The \( \phi^4 \) kink accelerates towards the source of the radiation
Oscillon state: $\phi^4$ model

(I. L. Bogolubsky and V. G. Makhankov (JETP Lett. 24, 12 (1976))

In the $\phi^4$ model there is a long lived nonradiative spatially localized solution (at least 10 million oscillations!!)

Gaussian initial data:

$$\phi(x, 0) = 1 - 0.7e^{-0.205x^2}$$

Collective coordinate model:

$$\phi(x, t) = 1 - A(t)e^{-\left(\frac{x}{x_0}\right)^2}$$

$$L/x_0 = (\dot{A})^2 - \frac{2}{3}A^4 - \pi A^3 - \left(4 + \frac{1}{3x_0^2}\right)A^2$$

Anharmonic oscillator with frequency $\Omega_0 = \sqrt{4 + \frac{1}{3x_0}}$
$\phi^4$ Kink-oscillon collisions

$v_{in} = 0.1$
$\phi^4$ Kink-oscillon collisions

$v_{in} = 0.2$
Sine-Gordon kink-breather collision

\[ v_{in} = 0.15 \]
$\phi^4$ $K\bar{K}$ collisions: fractal dynamics

D. Campbell, J. Schonfeld and C Wingate \textit{(Physica 9D (1983) 1)}

Annihilation:

$K\bar{K} \rightarrow$ oscillon

$\nu_{in} = 0.17$
\( \phi^4 K\bar{K} \) collisions: fractal dynamics

**Bounce:**

\[ K\bar{K} \rightarrow K\bar{K} \]

\( v_{in} = 0.27 \)
$\phi^4$ K$\bar{K}$ collisions: fractal dynamics

Three bounce resonance:

$K\bar{K} \rightarrow K\bar{K}$

$\nu_{in} = 0.24385$
Solitons vs. Solitary Waves

<table>
<thead>
<tr>
<th>Equation</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>S-G: $\ddot{\phi} - \phi'' + \sin \phi = 0$</td>
<td>YES $\phi_{K\bar{K}} = \pm 4 \arctan (e^{-x+x_0})$</td>
</tr>
<tr>
<td>$\lambda \phi^4$: $\ddot{\phi} - \phi'' - \phi + \phi^3 = 0$</td>
<td>NO! $\phi_{K\bar{K}} = \pm a \tanh \left( \frac{m(x-x_0)}{\sqrt{2}} \right)$</td>
</tr>
</tbody>
</table>

How do we know if it is integrable or it is a non-integrable?

Historically, combination of “experimental mathematics” ($\phi^4$) and known analytic solutions (S-G), then inverse scattering transform, group theoretic structure (Kac-Moody Algebras), Painlevé test.

Does any part of “hierarchy” of solitons in integrable theories (S-G breather) exist in non-integrable theories?
Consider a model with scalar field in $d$-dim

$$E[\phi] = \int d^d x \left[ \partial_\mu \phi \partial^\mu \phi + U(\phi) \right] = E_2 + E_0$$

**Scale transformation:**

$$\bar{x} \rightarrow \bar{y} = \lambda \bar{x}; \quad \partial_\mu \phi(\bar{x}) = \frac{\partial \phi(\bar{x})}{\partial x_\mu} \rightarrow \lambda \frac{\partial \phi(\lambda \bar{x})}{\partial (\lambda x_\mu)} = \lambda \frac{\partial \phi(\bar{y})}{\partial y_\mu}$$

$$d^d x \rightarrow d^d (\lambda x) \lambda^{-d} = \lambda^{-d} d^d y$$

$$E[\phi] \rightarrow \lambda^{2-d} E_2 + \lambda^{-d} E_0$$

Each term is positive. If there is a stationary point of $E(\lambda)$?

$$\frac{dE[\lambda \phi]}{d\lambda} = (2 - d) \lambda^{1-d} E_2 - d \lambda^{-d-1} E_0$$

![Graphs for different dimensions](d=1, d=2, d=3)
For a simple model

$$E[\phi] = \int d^d x \left[ \partial_\mu \phi \partial^\mu \phi + U(\phi) \right] = E_2 + E_0$$

nontrivial solutions \((E_2 \neq 0, E_0 \neq 0)\) are possible only in \(d=1\)

There are 3 possibilities to evade Derrick’s theorem:

• **\(d=2\)**: take \(E_0 = 0\), then the model is scale-invariant
  
  • Extend the model including higher derivatives in \(\phi\) (Skyrme model in \(d=3\), baby Skyrme model in \(d=2\), Faddeev-Skyrme model in \(d=3\))
  
  • Extend the model including gauge fields (monopoles in \(d=3\), instantons in Euclidean space \(d=4\))

\[ \begin{align*}
\bar{x} &\rightarrow \lambda \bar{x} = \bar{y}; \quad A_\mu(\bar{x}) \rightarrow \lambda A_\mu(\bar{y}); \quad D_\mu \phi(\bar{x}) \rightarrow \lambda D_\mu \phi(\bar{y}); \quad F_{\mu\nu}(\bar{x}) \rightarrow \lambda^2 F_{\mu\nu}(\bar{y}) \\
E[\phi] &= \int d^d x \left[ |F_{\mu\nu}|^2 + |D_\mu \phi|^2 + U(\phi) \right] = E_4 + E_2 + E_0 \\
E[\phi] \rightarrow \lambda^{4-d} E_4 + \lambda^{2-d} E_2 + \lambda^{-d} E_0
\end{align*} \]
If we restrict ourselves to the models with quadratic terms in derivatives, there are possibilities:

- **d=1**: there are soliton solutions in the models with gauge and scalar fields or in pure scalar models with a potential $U(\phi)$ (*Kinks*).
- **d=2**: there are soliton solutions in the models with gauge and scalar fields (*vortices*) or in pure scalar models without potential $U(\phi)$ (*Lumps*).
- **d=3**: there are soliton solutions in the models with gauge and scalar fields (*monopoles*).
- **d=4**: there are soliton solutions in the models with gauge field only (*instantons*).
- **d>4**: there are no soliton solutions, higher derivatives are necessary.

Alternative: one can consider time-dependent stationary configurations!